Geography of Non-Formal Symplectic and Contact Manifolds

Christoph Bock

MSC 1991: 57C15, 55S30, 55P62

Abstract

Let (m, b) be a pair of natural numbers. For m even (resp. m odd and $b \geq 2$) we show that if there is an m-dimensional non-formal compact oriented manifold with first Betti number $b_1 = b$, there is also a symplectic (resp. contact) manifold with these properties.

1 Introduction

In [7] Fernández and Muñoz proved the following:

Theorem 1.1. *Given $m \in \mathbb{N}_+$ and $b \in \mathbb{N}$, there are compact oriented m-dimensional manifolds with $b_1 = b$ which are non-formal if and only if one of the following conditions holds:*

(i) $m \geq 3$ *and* $b \geq 2$,

(ii) $m \geq 5$ *and* $b = 1$,

(iii) $m \geq 7$ *and* $b = 0$. □

A natural question to ask is when there are such examples of symplectic m-dimensional manifolds with $b_1 = b$ which are not formal. Clearly, m has to be even. The same authors as above constructed in [8] a simply-connected 8-dimensional example. Taking products with S^2, we get simply-connected examples for all even dimensions greater than eight.

Fernández, Gotay and Gray showed in [6] that there are T^2 bundles over T^2 which are symplectic, non-formal with $b_1 = 2, 3$ (and of course 4-dimensional). Again, by taking products with S^2, one also has examples for all even dimensions greater than four with the same b_1.

We prove:

Theorem 1.2. *For all $m \in 2\mathbb{N}$ and $b \in \mathbb{N}$, $m, b \geq 4$, there are compact m-dimensional symplectic manifolds with $b_1 = b$ which are non-formal.*

Theorem 1.3. *For all $m \in 2\mathbb{N}$, $m \geq 6$, there are compact m-dimensional symplectic manifolds with $b_1 = 1$ which are non-formal.*

These theorems and the considerations from above imply:

Theorem 1.4. *Let $(m, b) \in 2\mathbb{N}_+ \times \mathbb{N}$. If there is a non-formal compact oriented m-dimensional manifold with $b_1 = b$, then there is also a symplectic manifold with these properties.* □

Moreover, we will prove:

Theorem 1.5. *For each pair (m, b) with m odd and $b \geq 2$ there exists a non-formal compact contact m-manifold.*

2 Formality and s-Formality

We give a brief review of the notion of formality.

A *differential graded algebra (DGA)* is a graded $\mathbb{R}$-algebra $A = \bigoplus_{i\in\mathbb{N}} A^i$ together with an $\mathbb{R}$-linear map $d\colon A \to A$ such that $d(A^i) \subset A^{i+1}$ and the following conditions are satisfied:

(i) The $\mathbb{R}$-algebra structure of A is given by an inclusion $\mathbb{R} \hookrightarrow A^0$.

(ii) The multiplication is graded commutative, i.e. for $a \in A^i$ and $b \in A^j$ one has $a \cdot b = (-1)^{i\cdot j} b \cdot a \in A^{i+j}$.

(iii) The Leibniz rule holds: $\forall_{a\in A^i}\forall_{b\in A}\ d(a\cdot b) = d(a)\cdot b + (-1)^i a \cdot d(b)$

(iv) The map d is a differential, i.e. $d^2 = 0$.

Further, we define $|a| := i$ for $a \in A^i$.

The *i-th cohomology of a DGA* (A, d) is the algebra

$$H^i(A, d) := \frac{\ker(d\colon A^i \to A^{i+1})}{\operatorname{im}(d\colon A^{i-1} \to A^i)}.$$

If (B, d_B) is another DGA, then an $\mathbb{R}$-linear map $f\colon A \to B$ is called *morphism* if $f(A^i) \subset B^i$, f is multiplicative, and $d_B \circ f = f \circ d_A$. Obviously, any such f induces a homomorphism $f^*\colon H^*(A, d_A) \to H^*(B, d_B)$. A morphism of differential graded algebras inducing an isomorphism on cohomology is called *quasi-isomorphism.*

Definition 2.1. A DGA $(\mathcal{M}, d)$ is said to be *minimal* if

(i) there is a graded vector space $V = \left(\bigoplus_{i\in\mathbb{N}_+} V^i\right) = \operatorname{Span}\{a_k \mid k \in I\}$ with homogeneous elements a_k, which we call the generators,

(ii) $\mathcal{M} = \bigwedge V$,

(iii) the index set I is well ordered, such that $k < l \;\Rightarrow |a_k| \leq |a_l|$ and the expression for da_k contains only generators a_l with $l < k$.

We shall say that $(\mathcal{M}, d)$ is a *minimal model for a differential graded algebra* (A, d_A) if $(\mathcal{M}, d)$ is minimal and there is a quasi-isomorphism $\rho\colon (\mathcal{M}, d) \to (A, d_A)$, i.e. ρ induces an isomorphism $\rho^*\colon H^*(\mathcal{M}, d) \to H^*(A, d_A)$ on cohomology.

The importance of minimal models is reflected by the following theorem, which is taken from Sullivan's work [14, Section 5].

Theorem 2.2. *A differential graded algebra (A, d_A) with $H^0(A, d_A) = \mathbb{R}$ possesses a minimal model. It is unique up to isomorphism of differential graded algebras.*

A *minimal model $(\mathcal{M}_M, d)$ of a connected smooth manifold M* is a minimal model for the de Rahm complex $(\Omega(M), d)$ of differential forms on M. The last theorem implies that every connected smooth manifold possesses a minimal model which is unique up to isomorphism of differential graded algebras.

For the remainder of this section, we deal with the notion of formality. Endowed with the trivial differential, the cohomology of a minimal DGA is a DGA itself, and therefore it also possesses a minimal model. In general, these two minimal models need not to be isomorphic.

A minimal differential graded algebra $(\mathcal{M}, d)$ is called *formal* if there is a morphism of differential graded algebras

$$\psi\colon (\mathcal{M}, d) \longrightarrow (H^*(\mathcal{M}, d), d_H = 0)$$

that induces the identity on cohomology.

This means that $(\mathcal{M}, d)$ and $(H^*(\mathcal{M}, d), d_H = 0)$ share their minimal model. The following theorem gives an equivalent characterisation.

Theorem 2.3 ([13, Theorem 1.3.1]). *A minimal model $(\mathcal{M}, d)$ is formal if and only if we can write $\mathcal{M} = \bigwedge V$ and the space V decomposes as a direct sum $V = C \oplus N$ with $d(C) = 0$, d is injective on N, and such that every closed element in the ideal $I(N)$ generated by N in $\bigwedge V$ is exact.* □

This allows us to give a weaker version of the notion of formality.

Definition 2.4. A minimal model $(\mathcal{M}, d)$ is called *s-formal*, $s \in \mathbb{N}$, if we can write $\mathcal{M} = \bigwedge V$ and for each $i \leq s$ the space V^i generated by generators of degree i decomposes as a direct sum $V^i = C^i \oplus N^i$ with $d(C^i) = 0$, d is injective on N^i and such that every closed element in the ideal $I(\bigoplus_{i \leq s} N^i)$ generated by $\bigoplus_{i \leq s} N^i$ in $\bigwedge \left(\bigoplus_{i \leq s} V^i \right)$ is exact in $\bigwedge V$.

Obviously, formality implies s-formality for every s.

A connected smooth manifold is called *formal* (resp. *s-formal*) if its minimal model is formal (resp. s-formal).

We end this section with some results that allow an easier detection of formality resp. non-formality. The next theorem shows the reason for defining s-formality: in certain cases s-formality is sufficient for a manifold to be formal.

Theorem 2.5 ([9, Theorem 3.1]). *Let M be a connected and orientable compact smooth manifold of dimension $2n$ or $(2n - 1)$.*

Then M is formal if and only if it is $(n - 1)$-formal. □

Example 2.6 ([9, Corollary 3.3]).

(i) Every connected and simply-connected compact smooth manifold is 2-formal.

(ii) Every connected and simply-connected compact smooth manifold of dimension seven or eight is formal if and only if it is 3-formal. □

Proposition 2.7 ([9, Lemma 2.11]). *Let M_1, M_2 be connected smooth manifolds. They are both formal (resp. s-formal) if and only if $M_1 \times M_2$ is formal (resp. s-formal).* □

3 Massey Products

An important tool for detecting non-formality is the concept of Massey products: As we shall see below, the triviality of the Massey products is necessary for formality.

Let (A, d) be a differential graded algebra.

(i) Let $a_i \in H^{p_i}(A)$, $p_i > 0$, $1 \leq i \leq 3$, satisfying $a_j \cdot a_{j+1} = 0$ for $j = 1, 2$. Take elements α_i of A with $a_i = [\alpha_i]$ and write $\alpha_j \cdot \alpha_{j+1} = d\xi_{j,j+1}$ for $j = 1, 2$. The *(triple-)Massey product* $\langle a_1, a_2, a_3 \rangle$ of the classes a_i is defined as

$$[\alpha_1 \cdot \xi_{2,3} + (-1)^{p_1+1} \xi_{1,2} \cdot \alpha_3] \in \frac{H^{p_1+p_2+p_3-1}(A)}{a_1 \cdot H^{p_2+p_3-1}(A) + H^{p_1+p_2-1}(A) \cdot a_3}.$$

(ii) Now, let $k \geq 4$ and $a_i \in H^{p_i}(A)$, $p_i > 0$, $1 \leq i \leq k$, such that $\langle a_1, \ldots, a_{k-1} \rangle$ and $\langle a_2, \ldots, a_k \rangle$ are defined and vanish simultaneously, i.e. there are elements $\xi_{i,j}$ of A, $1 \leq i \leq j \leq k$, $(i, j) \neq (1, k)$, such that

$$a_i = [\xi_{i,i}] \quad \text{and} \quad d\xi_{i,j} = \sum_{l=i}^{j-1} \overline{\xi}_{i,l} \cdot \xi_{l+1,j}, \tag{1}$$

where $\overline{\xi} = (-1)^{|\xi|} \xi$. The *Massey product* $\langle a_1, \ldots, a_k \rangle$ of the classes a_i is defined as the set $\{[\sum_{l=1}^{k-1} \overline{\xi}_{1,l} \cdot \xi_{l+1,k}] \,|\, \xi_{i,j} \text{ satisfies (1)}\} \subset H^{p_1+\ldots+p_k-(k-2)}(A)$.

We say that $\langle a_1, \ldots, a_k \rangle$ vanishes if $0 \in \langle a_1, \ldots, a_k \rangle$.

Remark. The definition of the triple-Massey product in (i) as an element of a quotient space is well defined, see e.g. [13, Section 1.6].

The next two lemmata show the relation between formality (resp. s-formality) and Massey products.

Lemma 3.1 ([13, Theorem 1.6.5]). *For any formal minimal differential graded algebra all Massey products vanish.* □

Lemma 3.2 ([9, Lemma 2.9]). *Let (A, d) be an s-formal minimal differential graded algebra. Suppose that there are cohomology classes $a_i \in H^{p_i}(A)$, $p_i > 0$, $1 \leq i \leq k$, such that $\langle a_1, \ldots, a_k \rangle$ is defined. If $p_1 + \ldots + p_{k-1} \leq s + k - 2$ and $p_2 + \ldots + p_k \leq s + k - 2$, then $\langle a_1, \ldots, a_k \rangle$ vanishes.* □

In [8], Fernández and Muñoz introduce a different type of Massey product, called a-Massey product:

Definition 3.3. Let (A, d) be a DGA and let $a, b_1, b_2, b_3 \in H^2(A)$ satisfying $a \cdot b_i = 0$ for $i = 1, 2, 3$. Take choices of representatives $a = [\alpha], b_i = [\beta_i]$ and $\alpha \cdot \beta_i = d\xi_i$ for $i = 1, 2, 3$. Then the *a-Massey product* $\langle a; b_1, b_2, b_3 \rangle$ is defined as $[\xi_1 \cdot \xi_2 \cdot \beta_3 + \xi_2 \cdot \xi_3 \cdot \beta_1 + \xi_3 \cdot \xi_1 \cdot \beta_2]$ in

$$\frac{H^8(A)}{\langle b_1, a, b_2 \rangle \cdot H^3(A) + \langle b_1, a, b_3 \rangle \cdot H^3(A) + \langle b_2, a, b_3 \rangle \cdot H^3(A)}.$$

Lemma 3.4 ([8, Proposition 3.2]). *If a minimal differential graded algebra is formal, then every a-Massey product vanishes.* □

Corollary 3.5. *If the de Rahm complex $(\Omega(M), d)$ of a smooth manifold M possesses a non-vanishing Massey or a-Massey product, then M is not formal.*

If there are cohomology classes $a_i \in H^{p_i}(M)$, where $p_i > 0$ and $1 \leq i \leq k$, with $p_1 + \ldots + p_{k-1} \leq s + k - 2$ and $p_2 + \ldots + p_k \leq s + k - 2$ such that $\langle a_1, \ldots, a_k \rangle$ does not vanish, then M is not s-formal.

Proof. This holds since a minimal model $\rho\colon (\mathcal{M}_M, d) \to (\Omega(M), d)$ induces an isomorphism on cohomology. □

4 Donaldson Submanifolds

Our examples of non-formal symplectic manifolds will be constructed in a similar way as in the article [9] of Fernández and Muñoz. The examples will be Donaldson submanifolds of non-formal symplectic manifolds. Therefore, we quote in this section parts of [9].

For the remainder of the section we denote the de Rham cohomology of a smooth manifold M by $H^*(M)$.

In [5] the following is proven: Let (M, ω) be a $2n$-dimensional compact symplectic manifold with $[\omega] \in H^2(M)$ admitting a lift to an integral cohomology class. Then there exists $k_0 \in \mathbb{N}_+$ such that for each $k \in \mathbb{N}_+$ with $k \geq k_0$ there is a symplectic submanifold $j\colon Z \hookrightarrow M$ of dimension $2n-2$ whose Poincaré dual satisfies $\mathrm{PD}[Z] = k[\omega]$. Moreover, the map j is a homology $(n-1)$-equivalence in the following sense.

Let $f\colon M_1 \to M_2$ be a smooth map between smooth manifolds. f is called *homology s-equivalence*, $s \in \mathbb{N}$, if it induces isomorphisms $f^*\colon H^i(M_2) \to H^i(M_1)$ on cohomology for $i \leq s-1$ and a monomorphism for $i = s$.

A symplectic submanifold $j\colon Z \hookrightarrow M$ as above is called *symplectic divisor* or *Donaldson submanifold*.

Concerning minimal models and formality in this context, we quote the following results. Part (i) resp. (ii) in the theorem coincides with Proposition 5.1 resp. Theorem 5.2 (i) in [9], where a proof is given.

Theorem 4.1 ([9])**.** *Let $f\colon M_1 \to M_2$ be a homology s-equivalence between connected smooth manifolds. Denote by $\rho_i\colon (\bigwedge V_i, d) \to (\Omega(M_i), d)$ the minimal models of M_i for $i = 1, 2$.*

(i) There exist a morphism $F\colon (\bigwedge V_2^{\leq s}, d) \to (\bigwedge V_1^{\leq s}, d)$ of differential graded algebras such that $F\colon V_2^{<s} \to V_1^{<s}$ is an isomorphism, $F\colon V_2^s \to V_1^s$ is a monomorphism and $\rho_1^* \circ F^* = f^* \circ \rho_2^*$.

(ii) If M_2 is $(s-1)$-formal, then M_1 is $(s-1)$-formal. □

Corollary 4.2 ([9, Theorem 5.2(ii)])**.** *Let M be a $2n$-dimensional compact symplectic manifold and $j : Z \hookrightarrow M$ a Donaldson submanifold.*

Then for each $s \leq n-2$, we have: If M is s-formal, then Z is s-formal.

In particular, Z is formal if M is $(n-2)$-formal. □

Next, we want to give a criterion for a Donaldson submanifold not to be formal.

Proposition 4.3. *Let M be a compact symplectic manifold of dimension $2n$, where $n \geq 3$. Using the notation from page 3, we suppose that there are cohomology classes $a_i = [\alpha_i] \in H^1(M), 1 \leq i \leq 3$, such that the (triple-)Massey product*

$$\langle a_1, a_2, a_3\rangle = [\alpha_1 \wedge \xi_{2,3} + \xi_{1,2} \wedge \alpha_3] \in \frac{H^2(M)}{a_1 \cup H^1(M) + H^1(M) \cup a_3}$$

is defined and does not vanish.

Then every Donaldson submanifold of M is not 1-formal.

Proof. Let $j\colon Z \hookrightarrow M$ be a Donaldson submanifold. Since $n \geq 3$, j is a homology 2-equivalence. This implies that the (triple-)Massey product

$$\langle j^*a_1, j^*a_2, j^*a_3\rangle = [j^*\alpha_1 \wedge j^*\xi_{2,3} + j^*\xi_{1,2} \wedge j^*\alpha_3] \in \frac{H^2(Z)}{j^*a_1 \cup H^1(Z) + H^1(Z) \cup j^*a_3}$$

is defined and does not vanish. Now, Corollary 3.5 implies that Z is not 1-formal. □

As an immediate consequence of the proposition and its proof we get:

Corollary 4.4. *Let $Z_1, \ldots, Z_k, M$ be compact symplectic manifolds and assume that $Z_i \hookrightarrow Z_{i+1}$ and $Z_k \hookrightarrow M$ are Donaldson submanifolds for $i = 1, \ldots, k-1$. We suppose that there are cohomology classes $a_i = [\alpha_i] \in H^1(M), 1 \leq i \leq 3$, such that the (triple-)Massey product*

$$\langle a_1, a_2, a_3\rangle = [\alpha_1 \wedge \xi_{2,3} + \xi_{1,2} \wedge \alpha_3] \in \frac{H^2(M)}{a_1 \cup H^1(M) + H^1(M) \cup a_3}$$

is defined and does not vanish.

If $\dim Z_1 \geq 4$, then Z_1 is not 1-formal. □

The next lemma will be needed in the proof of Theorem 1.3. The proof is taken word by word from the proof of Formula (5) in [9].

Lemma 4.5. *Let (M, ω) be a $2n$-dimensional compact symplectic manifold and $j\colon Z \hookrightarrow M$ a Donaldson submanifold.*

Then for each $p = 2(n-1) - i$, $0 \leq i \leq (n-2)$, there is a monomorphism

$$\frac{H^p(M)}{\ker([\omega]\cup\colon H^p(M) \to H^{p+2}(M))} \longrightarrow H^p(Z).$$

Proof. The claim can be seen via Poincaré duality. Let $0 \leq i \leq (n-2)$, $p = 2(n-1) - i$ and $\alpha \in \Omega^p(M)$ be closed. Then we have

$$j^*[\alpha] = 0 \Longleftrightarrow \forall_{b \in H^i(Z)}\ j^*[\alpha] \cup b = 0.$$

Since $i \leq (n-2)$, we know that there is an isomorphism $j^*\colon H^i(M) \xrightarrow{\simeq} H^i(Z)$, thus we can assume that for each $b \in H^i(Z)$ there is a closed i-form β on M with $[\beta|_Z] = j^*[\beta] = b$ and get

$$j^*[\alpha] \cup j^*[\beta] = \int_Z j^*\alpha \wedge j^*\beta = \int_M \alpha \wedge \beta \wedge k\omega,$$

since $[Z] = \mathrm{PD}[k\omega]$ with $k \in \mathbb{N}_+$. Therefore, we have

$$j^*[\alpha] = 0 \Longleftrightarrow \forall_{[\beta] \in H^i(M)}\ [\alpha \wedge \omega] \cup [\beta] = 0 \Longleftrightarrow [\alpha \wedge \omega] = 0,$$

from where the lemma follows. □

5 Known Examples

5.1 The manifolds M(p,q)

The following examples are taken from [4].

Let R be a ring with 1. For $p \in \mathbb{N}_+$ let $H(1,p;R)$ be the set

$$\{\begin{pmatrix} I_p & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, z \in R^p,\ y \in R\}.$$

We write $H(1,p)$ for $H(1,p;\mathbb{R})$. Clearly, this is a nilpotent Lie group and the $2p+1$ differential 1-forms

$$\alpha_i := dx_i,\ \beta := dy,\ \gamma_i := dz_i - x_i\,dy,\ \ 1 \leq i \leq p$$

form a basis of the left-invariant 1-forms. Obviously, we have $d\alpha_i = d\beta = 0$ and $d\gamma_i = -\alpha_i \wedge \beta$.

Further, let $q \in \mathbb{N}_+$. We set $G(p,q) := H(1,p) \times H(1,q)$. Again, this is a Lie group and analogous as above, we denote the $2p+2q+2$ forms which form a basis of the left-invariant 1-forms by

$$\alpha_1, \ldots, \alpha_p, \beta, \gamma_1, \ldots, \gamma_p, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_q, \tilde{\beta}, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_q.$$

An easy computation shows that the 2-form

$$\omega := \sum_{i=1}^{p} \alpha_i \wedge \gamma_i + \sum_{i=1}^{q} \tilde{\alpha}_i \wedge \tilde{\gamma}_i + \beta \wedge \tilde{\beta}$$

is a left-invariant symplectic form. Therefore $M(p,q) := G(p,q)/\Gamma(p,q)$, where $\Gamma(p,q) := H(1,p;\mathbb{Z}) \times H(1,q;\mathbb{Z})$, is a compact symplectic nilmanifold of dimension $2p+2q+2$.

By [13, Theorem 2.1.3], the minimal model $\rho\colon \big(\mathcal{M}_{M(p,q)}, d\big) \to \big(\Omega(M(p,q)), d\big)$ is given by

$$\begin{gathered}
\mathcal{M}_{M(p,q)} = \bigwedge(a_1, \ldots, a_p, b, c_1, \ldots, c_p, \tilde{a}_1, \ldots, \tilde{a}_q, \tilde{b}, \tilde{c}_1, \ldots, \tilde{c}_q), \\
|a_i| = |b| = |c_i| = |\tilde{a}_i| = |\tilde{b}| = |\tilde{c}_i| = 1, \\
da_i = db = d\tilde{a}_i = d\tilde{b} = 0,\ dc_i = -a_i b,\ d\tilde{c}_i = -\tilde{a}_i \tilde{b}, \\
\rho(a_i) = \alpha_i,\ \rho(b) = \beta,\ \rho(c_i) = \gamma_i,\ \rho(\tilde{a}_i) = \tilde{\alpha}_i,\ \rho(\tilde{b}) = \tilde{\beta},\ \rho(\tilde{c}_i) = \tilde{\gamma}_i.
\end{gathered}$$

Therefore, we see $b_1(M(p,q)) = p + q + 2$.

Proposition 5.1 ([4]). *$M(p,q)$ is not formal.*

Proof. $\langle[\beta],[\alpha_i],[\alpha_i]\rangle$ is a non-vanishing Massey product. □

Using Theorem [13, Theorem 2.1.3] again, one computes the first cohomology groups of $M(p,q)$ as

$$\begin{aligned} H^0(M(p,q)) &= \langle 1\rangle, \\ H^1(M(p,q)) &= \langle[\alpha_i],[\beta],[\tilde\alpha_k],[\tilde\beta]\,|\,1\le i\le p, 1\le k\le q\rangle, \\ H^2(M(p,q)) &= \langle[\alpha_i\wedge\gamma_j],[\alpha_i\wedge\tilde\alpha_k],[\alpha_i\wedge\tilde\beta],[\beta\wedge\gamma_j],[\beta\wedge\tilde\alpha_l],[\beta\wedge\tilde\beta], \\ &\qquad [\tilde\alpha_k\wedge\tilde\gamma_l],[\tilde\beta\wedge\tilde\gamma_l]\,|\,1\le i,j\le p, 1\le k,l\le q\rangle. \end{aligned}$$

5.2 The manifold $\mathbf{M}_{8,0}$

Fernández and Muñoz constructed in [8] an 8-dimensional non-3-formal compact symplectic manifold $(M_{8,0},\omega)$ with

$$\begin{gathered} b_0(M_{8,0}) = b_8(M_{8,0}) = 1,\ b_1(M_{8,0}) = b_7(M_{8,0}) = 0, \\ b_2(M_{8,0}) = b_6(M_{8,0}) = 256,\ b_3(M_{8,0}) = b_5(M_{8,0}) = 0,\ b_4(M_{8,0}) = 269 \end{gathered} \tag{2}$$

as desingularisation of an orbifold. The latter is a $\mathbb{Z}_3$-quotient of a nilmanifold. The non-formality is proved by regarding the a-Massey product $\langle[\vartheta];[\tau_1],[\tau_2],[\tau_3]\rangle$ for certain closed 2-forms ϑ,τ_i on $M_{8,0}$: One has $\langle[\vartheta];[\tau_1],[\tau_2],[\tau_3]\rangle = \lambda\,[\omega^4]$ for $\lambda\neq 0$. Clearly, $\lambda\,\omega^4$ is not exact, and since $b_3(M_{8,0})=0$, it follows from Definition 3.3 that this a-Massey product does not vanish.

6 Proofs

6.1 Proof of Theorem 1.2

Because products with finitely many copies of S^2 give the higher-dimensional examples, it is enough to prove that for every $b\ge 4$ there is a non-formal compact symplectic 4-manifold M with $b_1(M)=b$.

Let $b\ge 4$ and choose $p,q\in\mathbb{N}_+$ such that $p+q+2=b$. Then $M(p,q)$ has dimension $2p+2q+2\ge 6$ and is a non-formal compact symplectic nilmanifold with $b_1(M(p,q))=b$ which satisfies the assumption of Corollary 4.4. Therefore, we get the required non-formal 4-manifold Z with $b_1(Z)=b_1(M(p,q))=b$. □

6.2 Proof of Theorem 1.3

Since direct products with finitely many copies of S^2 gives the higher-dimensional ones, it is enough to find a six-dimensional example. This is constructed in [1].

But using the ideas from above, one can construct an eight dimensional example as follows:

Gompf has shown in [11] that there is a compact symplectic 4-manifold $M_{4,1}$ with $b_1(M_{4,1})=1$. By Proposition 2.7, $M_{12,1} := M_{8,0}\times M_{4,1}$ is a compact symplectic 12-manifold which is not 3-formal. Clearly, we have $b_1(M_{12,1})=1$. Denote the projections by $\pi\colon M_{12,1}\to M_{8,0}$, $\mathfrak{p}\colon M_{12,1}\to M_{4,1}$ and the symplectic forms of $M_{8,0}, M_{4,1}$ and $M_{12,1}$ by ω,σ and $\Omega=\pi^*\omega+\mathfrak{p}^*\sigma$. Let ϑ,τ_i be the 2-forms of Section 5.2. We mentioned $\langle[\vartheta];[\tau_1],[\tau_2],[\tau_3]\rangle = \lambda\,[\omega^4]\neq 0$.

Let $j\colon Z_{10,1}\hookrightarrow M_{12,1}$ be a Donaldson submanifold. The 10-form

$$\Omega\wedge\lambda\,\pi^*\omega^4 = (\pi^*\omega+\mathfrak{p}^*\sigma)\wedge\lambda\,\pi^*\omega^4 = \lambda\,\mathfrak{p}^*\sigma\wedge\pi^*\omega^4$$

on $M_{12,1}$ does not represent the zero class in

$$H^{10}(M_{12,1}) \overset{(2)}{=} (\langle[\sigma^2]\rangle \otimes H^6(M_{8,0})) \oplus (H^2(M_{4,1}) \otimes H^8(M_{8,0})).$$

Therefore, we get from Lemma 4.5: $\lambda\, j^*\pi^*[\omega^4] \in H^8(Z_{10,1})\backslash\{0\}$. From (2) we know $H^5(M_{8,0}) = 0$. Hence $\langle[j^*\pi^*\tau_k], [j^*\pi^*\vartheta], [j^*\pi^*\tau_l]\rangle = 0$ for $1 \leq k, l \leq 3$. So in the following a-Massey product there is no indeterminacy:

$$\langle[j^*\pi^*\vartheta]; [j^*\pi^*\tau_1], [j^*\pi^*\tau_2], [j^*\pi^*\tau_3]\rangle = \lambda\, j^*\pi^*[\omega^4] \neq 0.$$

It follows that $Z_{10,1}$ is not formal. The fact that $\dim Z_{10,1} = 10$ and $b_1(Z_{10,1}) = 1$ is clear by the remarks in Section 4.

Now, let $\tilde{j}\colon Z_{8,1} \hookrightarrow Z_{10,1}$ be a Donaldson submanifold. Then the 10-form $j^*\Omega \wedge \lambda\, j^*\pi^*\omega^4$ on $Z_{10,1}$ does not represent the zero class in $H^{10}(Z_{10,1})$, for we have

$$\Omega^2 \wedge \pi^*\omega^4 = (\mathfrak{p}^*\sigma + \pi^*\omega) \wedge (\mathfrak{p}^*\sigma \wedge \pi^*\omega^4) = 2\,\mathfrak{p}^*\sigma^2 \wedge \pi^*\omega^4 \neq 0,$$

and by Lemma 4.5 we get $[j^*(\Omega \wedge \pi^*\omega^4)] \neq 0$.

Again we use Lemma 4.5 to see $\lambda\, \tilde{j}^*j^*\pi^*[\omega^4] \in H^8(Z_{8,1})\backslash\{0\}$ and can prove similarly as for $Z_{10,1}$ that $Z_{8,1}$ is not formal. Moreover, $Z_{8,1}$ is 8-dimensional and has first Betti number equal to one. $\square$

Remark. A Donaldson submanifold $Z_{6,1}$ of the manifold $Z_{8,1}$ that we have constructed in the last proof is formal: From the 2-formality of $M_{12,1} = M_{8,0} \times M_{4,1}$ it follows that $Z_{6,1}$ is 2-formal and therefore formal by Theorem 2.5.

6.3 Proof of Theorem 1.5

Our starting point is a non-formal symplectic manifold. Boothby and Wang proved that there is a contact manifold which fibres over it with fibre a circle.

Theorem 6.3.1 ([2, Theorem 3])**.** *If (M, ω) is a compact symplectic manifold whose symplectic form determines an integral cohomology class of M, then the principal circle bundle $\pi\colon E \to M$ with first Chern class $c_1(\pi) = [\omega] \in H^2(M, \mathbb{Z})$ admits a connection 1-form α such that $\pi^*\omega = d\alpha$ and α is a contact form on E.* $\square$

Let E, M be as in the last theorem. Since E is an S^1-bundle over M, one can apply the Gysin sequence to obtain $b_1(E) = b_1(M)$. We can even find a contact manifold which has the same fundamental group as M:

Corollary 6.3.2. *Let (M, ω) be a compact symplectic manifold of dimension $2n$ whose symplectic form determines an integral cohomology class.*

Then there is a compact contact manifold $(E, \ker\alpha)$ and a principal circle bundle $\pi\colon E \to M\#\overline{\mathbb{C}P}^n$ with first Chern class $c_1(\pi) = [\omega]$ such that the fundamental groups satisfy $\pi_1(E) = \pi_1(M\#\overline{\mathbb{C}P}^n) = \pi_1(M)$.

Proof. We use the same argumentation as in the proof of [11, Theorem 4.4]. After blowing up a point in M, we can obtain a manifold $M' := M\#\overline{\mathbb{C}P}^n$ with a symplectic form ω' such that $[\omega'] = [\omega] + \varepsilon e \in H^2(M') = H^2(M) \oplus H^2(\overline{\mathbb{C}P}^n)$, where $\varepsilon \in \frac{1}{\mathbb{N}_+}$ and e is a generator of $H^*(\overline{\mathbb{C}P}^n)$. Without loss of generality, we can assume that ω' determines an integral cohomology class and there is an embedded sphere $S \subset M' = M\#\overline{\mathbb{C}P}^n$ such that $\int_S \omega' = 1$. (Since $\int_S \omega'$ depends on the size of the ball removed from M in the blow-up, we may have to enlarge ω by an integer scale first.) Let $\pi\colon E \to M'$ with $c_1(\pi) = [\omega']$ as in Theorem 6.3.1. Then the restriction of the fibration π to S is the Hopf fibration, i.e. $\pi^{-1}(S) = S^3$ and the middle map in the following part of the homotopy sequence is an isomorphism:

$$\{0\} = \pi_2(\pi^{-1}(S)) \longrightarrow \pi_2(S) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(\pi^{-1}(S)) = \{1\}.$$

$\pi_2(S) \to \pi_1(S^1)$ is an isomorphism. From $S \subset M'$ we get in the following part of the homotopy sequence of the fibration π that the first map is surjective:

$$\pi_2(M') \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(E) \longrightarrow \pi_1(M') \longrightarrow \pi_0(S^1) = \{1\}.$$

This yields an isomorphism $\pi_*\colon \pi_1(E) \to \pi_1(M') = \pi_1(M)$. □

Under certain conditions we can show that the contact manifold that we have just constructed is not formal.

Proposition 6.3.3. *Let (M, ω) be a compact symplectic manifold of dimension $2n \geq 4$ whose symplectic form determines an integral cohomology class. Further, suppose that there are cohomology classes $a_i \in H^1(M)$, $1 \leq i \leq 3$, such that $\langle a_1, a_2, a_3\rangle$ is a non-vanishing Massey product in M.*

Then the manifold E of Corollary 6.3.2 is not formal.

Proof. Let $\pi\colon E \to M' := M\#\overline{\mathbb{C}P}^n$ be as in Corollary 6.3.2 and the non-vanishing Massey product $\langle a_1, a_2, a_3\rangle$ be defined by a 2-form $\alpha_1 \cdot \xi_{2,3} + \xi_{1,2} \cdot \alpha_3$. (Here we use the notation from page 3.) We show:

$$\pi^*\colon H^1(M') \to H^1(E) \text{ is an isomorphism.} \tag{3}$$

$$H^2(M) \cap \ker\big(\pi^*\colon H^2(M') \to H^2(E)\big) = \{0\} \tag{4}$$

Then $\pi^*\alpha_1 \cdot \pi^*\xi_{2,3} + \pi^*\xi_{1,2} \cdot \pi^*\alpha_3$ defines the non-vanishing Massey product

$$\begin{aligned} \langle \pi^*a_1, \pi^*a_2, \pi^*a_3\rangle \quad &\in \quad \frac{\pi^*\big(H^2(M')\big)}{\pi^*a_1 \cdot \pi^*\big(H^1(M')\big) + \pi^*\big(H^1(M')\big) \cdot \pi^*a_3} \\ &\subset \quad \frac{H^2(E)}{\pi^*a_1 \cdot H^1(E) + H^1(E) \cdot \pi^*a_3}, \end{aligned}$$

so E is not formal.

(Assume $\langle \pi^*a_1, \pi^*a_2, \pi^*a_3\rangle$ vanishes. Then for $j = 1, 2$ there exists a class $[\Xi_{j,j+1}] \in H^1(E)$ such that $0 = d\Xi_{j,j+1} = \pi^*\alpha_j \cdot \pi^*\alpha_{j+1}$. Property (3) implies the existence of $[\xi_{j,j+1}] \in H^1(M')$ with $0 = d\pi^*\xi_{j,j+1} = \pi^*\alpha_j \cdot \pi^*\alpha_{j+1}$ for $j = 1, 2$, i.e. $\alpha_j \cdot \alpha_{j+1}$ is exact by (4) and $\langle [\alpha_1], [\alpha_2], [\alpha_3]\rangle$ vanishes, which is a contradiction.)

It remains to show (3) and (4): Consider the Gysin sequence of π.

$$\{0\} \longrightarrow H^1(M') \xrightarrow{\pi^*} H^1(E) \longrightarrow H^0(M') \xrightarrow{[\omega']\cup} H^2(M') \xrightarrow{\pi^*} H^2(E) \longrightarrow \dots \tag{5}$$

Since $\cup[\omega]\colon H^0(M') \to H^2(M')$ is injective, it follows that $\pi^*\colon H^1(M') \to H^1(E)$ is an isomorphism.

Further, we get $\ker\big(\pi^*\colon H^2(M') \to H^2(E)\big) \overset{(5)}{=} \mathbb{R}[\omega']$. Denote

$$\mathfrak{pr}_2\colon H^2(M') = H^2(M) \oplus H^2(\overline{\mathbb{C}P}^n) \longrightarrow H^2(\overline{\mathbb{C}P}^n)$$

the projection onto the second factor. Since ω' is the symplectic form of the blow-up of M, we have $\mathfrak{pr}_2([\omega']) \neq 0$. But $\mathfrak{pr}_2|_{H^2(M)} = 0$, so (4) follows. □

Using the preparations that we have done, we are able to construct explicit non-formal contact manifolds.

Theorem 6.3.4. *For each $n \in \mathbb{N}$ with $n \geq 2$ and $b \in \{2, 3\}$ there exists a compact contact $(2n + 1)$-manifold which is not formal.*

Proof. In [6] the following manifolds are studied. Let M_b, $b \in \{2,3\}$ be the four-dimensional nilmanifold with basis of left-invariant 1-forms $\{\alpha,\beta,\gamma,\eta\}$ such that

$$\begin{aligned} d\alpha &= d\beta = 0, \\ d\gamma &= \alpha \wedge \beta, \\ d\eta &= \begin{cases} \alpha\wedge\gamma, & \text{if } b = 2, \\ 0, & \text{if } b = 3. \end{cases} \end{aligned}$$

Then, $b_1(M_b) = b$, the 2-form $\alpha \wedge \eta + \beta \wedge \gamma$ is a symplectic form for M_b, and $\langle [\beta],[\beta],[\alpha]\rangle = -[\beta \wedge \gamma]$ is a non-vanishing Massey product. The case $n = 2$ now follows from Proposition 6.3.3. For $n > 2$ consider the manifolds $M_b \times (S^2)^{n-2}$ instead of M_b. □

Theorem 6.3.5. *For each $b \in \mathbb{N}$ with $b \geq 2$, there are non-formal compact contact manifolds of dimension 3 and 5 with first Betti number $b_1 = b$.*

Proof. By Theorem 1.1, we know that there is a compact oriented 3-manifold M with $b_1 = b \geq 2$ which is non-formal. By theorems of Martinet [12] and Geiges [10, Proposition 2] M and $M \times S^2$ admit contact structures. Further, it follows from Proposition 2.7 that $M \times S^2$ is not formal. □

Now, Theorem 1.5 follows from Theorems 6.3.4, 6.3.5, Proposition 2.7 and the following result of Bourgeois:

Theorem 6.3.6 ([3])**.** *Let M be a compact contact manifold of dimension greater than or equal to three.*

Then $M \times T^2$ admits a contact structure. □

Note that the case $(2n+1, b)$ in Theorem 1.5 is realized if $(2n-1, b-2)$ is realized. Inductively, one gets to either the case $b \in \{2,3\}$, $2n+1 \geq 3$ or the case $b \geq 4$, $2n+1 = 3$, both covered previously. □

Acknowledgement. The results presented in this paper were obtained in my dissertation under the supervision of Prof. H. Geiges. I wish to express my sincerest gratitude for his support during the last four years.

References

[1] Ch. Bock: *On low-dimensional solvmanifolds.* in preparation

[2] W. M. Boothby, H. C. Wang: *On contact manifolds.* Ann. of Math. (2) **68** (1958), 721–734

[3] F. Bourgeois: *Odd dimensional tori are contact manifolds.* Int. Math. Res. Notices **30** (2002), 1571–1574

[4] L. A. Cordero, M. Fernández, A. Gray: *Symplectic manifolds with no Kähler structure.* Topology **25** (1986), no. 3, 375–380

[5] S. K. Donaldson: *Symplectic submanifolds and almost-complex geometry.* J. Differential Geom. **44** (1996), no. 4, 666–705

[6] M. Fernández, M. J. Gotay, A. Gray: *Compact parallelizable four dimensional symplectic and complex manifolds.* Proc. Amer. Math. Soc. **103** (1988), no. 4, 1209–1212

[7] M. Fernández, V. Muñoz: *The geography of non-formal manifolds.*
In: O. Kowalski, E. Musso, D. Perrone: *Complex, Contact and Symmetric Manifolds*, pp. 121–129. Prog. Math. vol. 234, Birkhäuser 2005

[8] M. Fernández, V. Muñoz: *An 8-dimensional non-formal simply connected symplectic manifold.*
Ann. of Math. (2) **167** (2008), no. 3, 1045–1054

[9] M. Fernández, V. Muñoz: *Formality of Donaldson submanifolds.*
Math. Z. **250** (2005), no. 1, 149–175

[10] H. Geiges: *Constructions of contact manifolds.*
Math. Proc. Cambridge Philos. Soc. **121** (1997), no. 3, 455–464

[11] R. Gompf: *A new construction of symplectic manifolds.*
Ann. of Math. (2) **142** (1995), no. 3, 527–595

[12] J. Martinet: *Formes de contact sur les variétés de dimension* 3.
In: Ch. T. C. Wall: *Proc. Liverpool Singularities Sympos. II*, pp. 142–163. Lecture Notes in Math. **209**, Springer 1971

[13] J. Oprea, A. Tralle: *Symplectic Manifolds with no Kähler Structure.*
Lecture Notes in Math. **1661**, Springer 1997

[14] D. Sullivan: *Infinitesimal computations in topology.*
Inst. Hautes Études Sci. Publ. Math. **47** (1977), 269–331

Christoph Bock
Universität zu Köln
Mathematisches Institut
Weyertal 86–90
D-50931 Köln
e-mail: bock@math.uni-koeln.de